

# PROPAGATOR OF A CHARGED PARTICLE WITH A SPIN IN UNIFORM MAGNETIC AND PERPENDICULAR ELECTRIC FIELDS

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*Dedicated to the memory of Professor Basil Nicolaenko*

**ABSTRACT.** We construct an explicit solution of the Cauchy initial value problem for the time-dependent Schrödinger equation for a charged particle with a spin moving in a uniform magnetic field and a perpendicular electric field varying with time. The corresponding Green function (propagator) is given in terms of elementary functions and certain integrals of the fields with a characteristic function, which should be found as an analytic or numerical solution of the equation of motion for the classical oscillator with a time-dependent frequency. We discuss a particular solution of a related nonlinear Schrödinger equation and some special and limiting cases are outlined.

## 1. INTRODUCTION

The time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H(t) \psi \quad (1.1)$$

can be solved using the time evolution operator given formally by

$$U(t, t_0) = T \left( \exp \left( -\frac{i}{\hbar} \int_{t_0}^t H(t') dt' \right) \right), \quad (1.2)$$

where  $T$  is the time ordering operator which orders operators with larger times to the left [3], [12]. This unitary operator takes a state at time  $t_0$  to a state at time  $t$ , so that

$$\psi(x, t) = U(t, t_0) \psi(x, t_0). \quad (1.3)$$

The simplicity of these formulas is deceptive, since the time evolution operator can be found explicitly as certain integral operator only in a few special cases. An important example of this source is the forced harmonic oscillator originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics [7], [8], [9], [10], and [11]; see also [21]. Since then this problem and its special and limiting cases were discussed by many authors; see Ref. [4], [13], [16], [22], [24], [39] for the simple harmonic oscillator and Ref. [2], [6], [15], [27], [33] for the particle in a constant external field and references therein. It is worth noting that an exact solution of the  $n$ -dimensional time-dependent Schrödinger equation for certain modified oscillator is found in [23].

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In this Letter we construct the time evolution operator explicitly in a general case of the one-dimensional Schrödinger equation when the Hamiltonian is an arbitrary quadratic form of the operator of coordinate and the operator of linear momentum; see equation (2.1) below. In this approach, all exactly solvable models, that have been cited above, are classified in terms of elementary solutions of a characterization equation given by (2.13) below. A particular solution of the corresponding nonlinear Schrödinger equation is obtained in a similar fashion. By separation of variables, we apply this method to another classical problem — the motion of a charged particle with a spin in uniform magnetic and perpendicular electric fields that are changing with time. The corresponding Green function (or Feynman's propagator) is found as an elementary function of certain integrals of our characteristic function and the electromagnetic fields; see equation (6.14) below. Special cases of constant and linear magnetic fields are discussed as examples. Moreover, these explicit solutions can also be useful when testing numerical methods of solving the time-dependent Schrödinger equation.

## 2. SOLUTION OF A CAUCHY INITIAL VALUE PROBLEM

The fundamental solution of the time-dependent Schrödinger equation with the quadratic Hamiltonian of the form

$$i \frac{\partial \psi}{\partial t} = -a(t) \frac{\partial^2 \psi}{\partial x^2} + b(t) x^2 \psi - i \left( c(t) x \frac{\partial \psi}{\partial x} + d(t) \psi \right) - f(t) x \psi + i g(t) \frac{\partial \psi}{\partial x}, \quad (2.1)$$

where  $a(t)$ ,  $b(t)$ ,  $c(t)$ ,  $d(t)$ ,  $f(t)$ , and  $g(t)$  are given real-valued functions of time  $t$  only, can be found with the help of a familiar substitution

$$\psi = A e^{iS} = A(t) e^{iS(x,y,t)} \quad (2.2)$$

with

$$A = A(t) = \frac{1}{\sqrt{2\pi i \mu(t)}} \quad (2.3)$$

and

$$S = S(x, y, t) = \alpha(t) x^2 + \beta(t) xy + \gamma(t) y^2 + \delta(t) x + \varepsilon(t) y + \kappa(t), \quad (2.4)$$

where  $\alpha(t)$ ,  $\beta(t)$ ,  $\gamma(t)$ ,  $\delta(t)$ ,  $\varepsilon(t)$ , and  $\kappa(t)$  are differentiable real-valued functions of time  $t$  only. Indeed,

$$\frac{\partial S}{\partial t} = -a \left( \frac{\partial S}{\partial x} \right)^2 - bx^2 + fx + (g - cx) \frac{\partial S}{\partial x} \quad (2.5)$$

by choosing

$$\frac{\mu'}{2\mu} = a \frac{\partial^2 S}{\partial x^2} + d = 2\alpha(t) a(t) + d(t). \quad (2.6)$$

Equating the coefficients of all admissible powers of  $x^m y^n$  with  $0 \leq m + n \leq 2$ , gives the following system of ordinary differential equations

$$\frac{d\alpha}{dt} + b(t) + 2c(t) \alpha + 4a(t) \alpha^2 = 0, \quad (2.7)$$

$$\frac{d\beta}{dt} + (c(t) + 4a(t) \alpha(t)) \beta = 0, \quad (2.8)$$

$$\frac{d\gamma}{dt} + a(t) \beta^2(t) = 0, \quad (2.9)$$

$$\frac{d\delta}{dt} + (c(t) + 4a(t)\alpha(t))\delta = f(t) + 2\alpha(t)g(t), \quad (2.10)$$

$$\frac{d\varepsilon}{dt} = (g(t) - 2a(t)\delta(t))\beta(t), \quad (2.11)$$

$$\frac{d\kappa}{dt} = g(t)\delta(t) - a(t)\delta^2(t), \quad (2.12)$$

where the first equation is the familiar Riccati nonlinear differential equation; see, for example, [14], [32], [40] and references therein. Substitution of (2.6) into (2.7) results in the second order linear equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \quad (2.13)$$

with

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \quad \sigma(t) = ab - cd + d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right), \quad (2.14)$$

which must be solved subject to the initial data

$$\mu(0) = 0, \quad \mu'(0) = 2a(0) \neq 0 \quad (2.15)$$

in order to satisfy the initial condition for the corresponding Green function; see the asymptotic formula (2.24) below. We shall refer to equation (2.13) as the *characteristic equation* and its solution  $\mu(t)$ , subject to (2.15), as the *characteristic function*. As the special case (2.13) contains the generalized equation of hypergeometric type, whose solutions are studied in detail in [29]; see also [1], [28], [38], and [40].

Thus, the Green function (fundamental solution or propagator) is explicitly given in terms of the characteristic function

$$\psi = G(x, y, t) = \frac{1}{\sqrt{2\pi i \mu(t)}} e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t))}. \quad (2.16)$$

Here

$$\alpha(t) = \frac{1}{4a(t)} \frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a(t)}, \quad (2.17)$$

$$\beta(t) = -\frac{1}{\mu(t)} \exp \left( -\int_0^t (c(\tau) - 2d(\tau)) d\tau \right), \quad (2.18)$$

$$\begin{aligned} \gamma(t) = & \frac{a(t)}{\mu(t)\mu'(t)} \exp \left( -2 \int_0^t (c(\tau) - 2d(\tau)) d\tau \right) \\ & - 4 \int_0^t \frac{a(\tau)\sigma(\tau)}{(\mu'(\tau))^2} \left( \exp \left( -2 \int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda \right) \right) d\tau, \end{aligned} \quad (2.19)$$

$$\begin{aligned} \delta(t) = & \frac{1}{\mu(t)} \exp \left( -\int_0^t (c(\tau) - 2d(\tau)) d\tau \right) \int_0^t \exp \left( \int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda \right) \\ & \times \left( \left( f(\tau) - \frac{d(\tau)}{a(\tau)} g(\tau) \right) \mu(\tau) + \frac{g(\tau)}{2a(\tau)} \mu'(\tau) \right) d\tau, \end{aligned} \quad (2.20)$$

$$\varepsilon(t) = -\frac{2a(t)}{\mu'(t)} \delta(t) \exp \left( -\int_0^t (c(\tau) - 2d(\tau)) d\tau \right) \quad (2.21)$$

$$\begin{aligned}
& +8 \int_0^t \frac{a(\tau) \sigma(\tau)}{(\mu'(\tau))^2} \exp \left( - \int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda \right) (\mu(\tau) \delta(\tau)) d\tau \\
& +2 \int_0^t \frac{a(\tau)}{\mu'(\tau)} \exp \left( - \int_0^\tau (c(\lambda) - 2d(\lambda)) d\lambda \right) \left( f(\tau) - \frac{d(\tau)}{a(\tau)} g(\tau) \right) d\tau, \\
\kappa(t) &= \frac{a(t) \mu(t)}{\mu'(t)} \delta^2(t) - 4 \int_0^t \frac{a(\tau) \sigma(\tau)}{(\mu'(\tau))^2} (\mu(\tau) \delta(\tau))^2 d\tau \\
& -2 \int_0^t \frac{a(\tau)}{\mu'(\tau)} (\mu(\tau) \delta(\tau)) \left( f(\tau) - \frac{d(\tau)}{a(\tau)} g(\tau) \right) d\tau
\end{aligned} \tag{2.22}$$

with

$$\delta(0) = \frac{g(0)}{2a(0)}, \quad \varepsilon(0) = -\delta(0), \quad \kappa(0) = 0. \tag{2.23}$$

We have used integration by parts to resolve the singularities of the initial data. Then the corresponding asymptotic formula is

$$G(x, y, t) = \frac{e^{iS(x, y, t)}}{\sqrt{2\pi i \mu(t)}} \rightarrow \frac{1}{\sqrt{4\pi i a(0)t}} \exp \left( i \frac{(x-y)^2}{4a(0)t} \right) \exp \left( i \frac{g(0)}{2a(0)} (x-y) \right) \tag{2.24}$$

as  $t \rightarrow 0^+$ . Notice that the first term on the right hand side is a familiar free particle propagator (cf. (3.1) below).

By the superposition principle, we obtain an explicit solution of the Cauchy initial value problem

$$i \frac{\partial \psi}{\partial t} = H(t) \psi, \quad \psi(x, t)|_{t=0} = \psi_0(x) \tag{2.25}$$

on the infinite interval  $-\infty < x < \infty$  with the general quadratic Hamiltonian as in (2.1) in the form

$$\psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \psi_0(y) dy. \tag{2.26}$$

This yields the time evolution operator (1.2) explicitly as an integral operator. Properties of similar oscillatory integrals are discussed in [36].

### 3. SOME SPECIAL CASES

Now let us consider several elementary solutions of the characteristic equation (2.13); more complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [1], [29], [31], and [40]. Among important elementary cases of our general expressions for the Green function (2.16)–(2.22) are the following:

When  $a = \hbar/2m$ ,  $b = c = d = f = g = 0$ , and  $\mu'' = 0$ ,  $\mu = t\hbar/m$ , one gets

$$G(x, y, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left( \frac{im(x-y)^2}{2\hbar t} \right), \tag{3.1}$$

which is the free particle propagator [11].

For a particle in a constant external field, where  $a = \hbar/2m$ ,  $b = c = d = g = 0$  and  $f = F/\hbar = \text{constant}$ ,  $\mu = t\hbar/m$ , the propagator is given by

$$G(x, y, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{im(x-y)^2}{2\hbar t}\right) \exp\left(\frac{iF(x+y)}{2\hbar}t - \frac{iF^2}{24\hbar m}t^3\right). \quad (3.2)$$

This case was studied in detail in [2], [6], [11], [15], [27] and [33]. Here we have corrected a typo in [11]; see [37] for a complete list of known errata in the Feynman and Hibbs book.

The simple harmonic oscillator with  $a = \hbar/2m$ ,  $b = m\omega^2/2\hbar$ ,  $c = d = f = g = 0$  and  $\mu'' + \omega^2\mu = 0$ ,  $\mu = (\hbar/m\omega) \sin \omega t$  has the familiar propagator of the form

$$G(x, y, t) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega t}} \exp\left(\frac{im\omega}{2\hbar \sin \omega t} ((x^2 + y^2) \cos \omega t - 2xy)\right), \quad (3.3)$$

which is studied in detail at [4], [13], [16], [22], [24], [39]. For an extension to the case of the forced harmonic oscillator including an extra velocity-dependent term and a time-dependent frequency, see [7], [8], [11] and [21].

Furthermore, an exact solution of the  $n$ -dimensional time-dependent Schrödinger equation for certain modified oscillator is found in [23]. In the one-dimensional case we get functions

$$a = \frac{1}{2}(1 + \cos 2t), \quad b = \frac{1}{2}(1 - \cos 2t), \quad c = 2d = \sin 2t \quad (3.4)$$

and our characteristic equation (2.13) takes the form

$$\mu'' + 2 \tan t \mu' - 2\mu = 0, \quad (3.5)$$

whose elementary solution is

$$\mu = \cos t \sinh t + \sin t \cosh t, \quad (3.6)$$

which satisfies the initial conditions (2.15). Further, the corresponding propagator is given by

$$G(x, y, t) = \frac{1}{\sqrt{2\pi i (\cos t \sinh t + \sin t \cosh t)}} \times \exp\left(\frac{(x^2 - y^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2i (\cos t \sinh t + \sin t \cosh t)}\right), \quad (3.7)$$

which was found in [23] as the special case  $n = 1$  of a general  $n$ -dimensional expansion of the Green function in hyperspherical harmonics. We have showed that (3.7) is a generalization of the propagator for the simple harmonic oscillator; see Ref. [23] for more details.

#### 4. A PARTICULAR SOLUTION OF THE NONLINEAR SCHRÖDINGER EQUATION

The method of solving the equation (2.1) can be extended to the nonlinear Schrödinger equation with a general quadratic Hamiltonian of the form

$$\begin{aligned} i \frac{\partial \psi}{\partial t} = & -a(t) \frac{\partial^2 \psi}{\partial x^2} + b(t) x^2 \psi - i \left( c(t) x \frac{\partial \psi}{\partial x} + d(t) \psi \right) \\ & - f(t) x \psi + i g(t) \frac{\partial \psi}{\partial x} + h(t) |\psi|^{2s} \psi, \quad s \geq 0, \end{aligned} \quad (4.1)$$

where  $a(t)$ ,  $b(t)$ ,  $c(t)$ ,  $d(t)$ ,  $f(t)$ ,  $g(t)$ , and  $h(t)$  are certain functions of time  $t$  only. A substitution of

$$\psi = \psi(x, t) = \frac{e^{i\phi}}{\sqrt{\mu(t)}} e^{iS(x, y, t)}, \quad \phi = \text{constant}, \quad (4.2)$$

where  $S = S(x, y, t) = \alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)$  with the same relation (2.6), results in a modified system (2.7)–(2.12) below:

$$\frac{d\alpha}{dt} + b(t) + 2c(t)\alpha + 4a(t)\alpha^2 = 0, \quad (4.3)$$

$$\frac{d\beta}{dt} + (c(t) + 4a(t)\alpha(t))\beta = 0, \quad (4.4)$$

$$\frac{d\gamma}{dt} + a(t)\beta^2(t) = 0, \quad (4.5)$$

$$\frac{d\delta}{dt} + (c(t) + 4a(t)\alpha(t))\delta = f(t) + 2\alpha(t)g(t), \quad (4.6)$$

$$\frac{d\varepsilon}{dt} = (g(t) - 2a(t)\delta(t))\beta(t), \quad (4.7)$$

$$\frac{d\kappa}{dt} = g(t)\delta(t) - a(t)\delta^2(t) - \frac{h(t)}{\mu^s(t)}, \quad (4.8)$$

where only the last equation has an extra term which corresponds to the nonlinear term in the original Schrödinger equation (4.1). Therefore equations (2.13)–(2.16) solve once again the Riccati equation (4.3), however; in this case, we would like to use nonsingular initial conditions

$$\mu(0) \neq 0, \quad \mu'(0) = 2(2\alpha(0)a(0) + d(0))\mu(0). \quad (4.9)$$

With these conditions, our system (4.3)–(4.8) can be solved again in terms of the characteristic function  $\mu(t)$ , thus giving us a particular solution of the nonlinear Schrödinger equation (4.1) in the form (4.2), corresponding to the initial data

$$\psi_0(x) = \psi(x, t)|_{t=0} = \frac{e^{i\phi}}{\sqrt{\mu(0)}} e^{i(\alpha(0)x^2 + \beta(0)xy + \gamma(0)y^2 + \delta(0)x + \varepsilon(0)y + \kappa(0))}. \quad (4.10)$$

The details here are left to the reader.

In the simplest case,

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + h|\psi|^{2s}\psi, \quad h = \text{constant}, \quad s \geq 0, \quad (4.11)$$

the solution of the characteristic equation  $\mu'' = 0$  is  $\mu(t) = \mu_0 + t\mu_1$ ,  $\mu_0 > 0$  and the coefficients of the quadratic form are given by

$$\alpha(t) = \frac{\mu_1}{2(\mu_0 + t\mu_1)}, \quad \beta(t) = \frac{\mu_0\beta_0}{\mu_0 + t\mu_1}, \quad \delta(t) = \frac{\mu_0\delta_0}{\mu_0 + t\mu_1}, \quad (4.12)$$

$$\gamma(t) = \gamma_0 - \frac{\mu_0\beta_0^2 t}{2(\mu_0 + t\mu_1)}, \quad \varepsilon(t) = \varepsilon_0 - \frac{\mu_0\beta_0\delta_0 t}{\mu_0 + t\mu_1}, \quad (4.13)$$

$$\kappa(t) = \kappa_0 - \frac{\mu_0\delta_0^2 t}{2(\mu_0 + t\mu_1)} - \frac{h}{\mu_1}\xi_s(t) \quad (4.14)$$

with

$$\xi_s(t) = \begin{cases} \frac{1}{(1-s)} ((\mu_0 + t\mu_1)^{1-s} - \mu_0^{1-s}), & \text{when } s \neq 1, \\ \ln \left( 1 + \frac{t\mu_1}{\mu_0} \right), & \text{when } s = 1. \end{cases} \quad (4.15)$$

Now, the limiting case where  $\mu_1 \rightarrow 0$  with  $\mu_0 > 0$  is given by

$$\psi = \frac{1}{\sqrt{\mu_0}} e^{i(\beta_0 xy + (\gamma_0 - \beta_0^2 t/2)y^2 + \delta_0 x + (\varepsilon_0 - \beta_0 \delta_0 t)y + \kappa_0 - \delta_0^2 t/2 - ht/\mu_0^s)}. \quad (4.16)$$

Then  $|\psi| = 1/\sqrt{\mu_0}$  is bounded at all times. Yet, when  $\mu_1 \neq 0$ , one gets

$$|\psi(x, t)| = \frac{1}{\sqrt{\mu_0 + t\mu_1}}, \quad t \geq 0, \quad (4.17)$$

which is bounded if  $\mu_0 > 0$  and  $\mu_1 > 0$ , and blows up at a finite time  $t_0 = -\mu_0/\mu_1$  if  $\mu_1 < 0$ .

The same method shows that the Cauchy initial value problem

$$\begin{aligned} i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} &= h |\psi|^{2s} \psi, \quad h = \text{constant}, \quad s \geq 0, \\ \psi|_{t=0} &= \delta_\varepsilon(x - y) = \frac{1}{\sqrt{2\pi i \varepsilon}} \exp\left(\frac{i(x - y)^2}{2\varepsilon}\right), \quad \varepsilon > 0 \end{aligned} \quad (4.18)$$

has the classical solution of the form

$$\psi = G_\varepsilon(x, y, t) = \frac{1}{\sqrt{2\pi i(t + \varepsilon)}} \exp\left(\frac{i(x - y)^2}{2(t + \varepsilon)} - \frac{ih}{2\pi} \chi_s(t)\right), \quad (4.19)$$

where

$$\chi_s(t) = \begin{cases} \frac{(t + \varepsilon)^{1-s} - \varepsilon^{1-s}}{1 - s}, & \text{when } s \neq 1, \\ \ln\left(1 + \frac{t}{\varepsilon}\right), & \text{when } s = 1 \end{cases} \quad (4.20)$$

with  $\chi_s(0) = 0$ . Here

$$\psi|_{t=0} = G_\varepsilon(x, y, 0) = \delta_\varepsilon(x - y) \rightarrow \delta(x - y) \quad (4.21)$$

as  $\varepsilon \rightarrow 0^+$  in the distributional sense

$$\lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} G_\varepsilon(x, y, 0) \varphi(y) dy = \varphi(x). \quad (4.22)$$

Further details are left to the reader.

In a similar fashion, the nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = -\cos^2 t \frac{\partial^2 \psi}{\partial x^2} + \sin^2 t x^2 \psi - i \sin t \cos t \left( 2x \frac{\partial \psi}{\partial x} + \psi \right) + 2 \cos t \sinh t |\psi|^{2s} \psi \quad (4.23)$$

(cf. [23]) has a particular solution of the form

$$\psi = \psi(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{i(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \kappa(t))}, \quad (4.24)$$

where  $\mu(t) = \cos t \cosh t + \sin t \sinh t$  and the coefficients are given by

$$\alpha(t) = \frac{\cos t \sinh t - \sin t \cosh t}{2(\cos t \cosh t + \sin t \sinh t)}, \quad (4.25)$$

$$\beta(t) = \frac{1}{\cos t \cosh t + \sin t \sinh t}, \quad (4.26)$$

$$\gamma(t) = -\frac{\cos t \sinh t + \sin t \cosh t}{2(\cos t \cosh t + \sin t \sinh t)}, \quad (4.27)$$

$$\kappa(t) = \begin{cases} -\frac{(\cos t \cosh t + \sin t \sinh t)^{1-s} - 1}{1-s}, & \text{when } s \neq 1, \\ -\ln(\cos t \cosh t + \sin t \sinh t), & \text{when } s = 1. \end{cases} \quad (4.28)$$

Thus, initial function is the standing wave

$$\psi_0(x) = \psi(x, t)|_{t=0} = e^{ixy}. \quad (4.29)$$

These exact solutions may be of interest in a general treatment of the nonlinear Schrödinger equation (see [17], [19], [26], [34], [35], [41] and references therein).

## 5. MOTION IN PERPENDICULAR MAGNETIC AND ELECTRIC FIELDS

A particle with a spin  $s$  also has a magnetic momentum  $\boldsymbol{\mu}$  with the operator

$$\hat{\boldsymbol{\mu}} = \mu \hat{\mathbf{s}}/s, \quad (5.1)$$

where  $\hat{\mathbf{s}}$  is the spin operator and  $\mu$  is a constant characterizing the particle. For the motion of a charged particle in a uniform magnetic field  $\mathbf{H}$  and an electric field  $\mathbf{E}$ , both which are perpendicular to each other, the three-dimensional time-dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \quad (5.2)$$

has a Hamiltonian operator as in [20], namely,

$$\hat{H} = \frac{1}{2m} \left( \hat{p}_x + \frac{eH}{c} y \right)^2 + \frac{1}{2m} \hat{p}_y^2 + \frac{1}{2m} \hat{p}_z^2 - \frac{\mu}{s} \hat{s}_z H - yF, \quad (5.3)$$

where  $\hat{\mathbf{p}} = -i\hbar \nabla$  is the linear momentum operator. The corresponding vector and scalar potentials are defined up to the gauge transformation. We use the original choice [20] for the vector potential  $\mathbf{A} = -yH(t) \mathbf{e}_x$  and add a linear scalar potential  $\varphi = -(F(t)/e)y$  (see also [21]). Then the uniform magnetic field  $\mathbf{H}$  and the corresponding perpendicular electric field  $\mathbf{E}$  are given by

$$\mathbf{H} = \text{curl } \mathbf{A} = H(t) \mathbf{e}_z, \quad \mathbf{E} = -\nabla \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{y}{c} \frac{dH(t)}{dt} \mathbf{e}_x + \frac{F(t)}{e} \mathbf{e}_y; \quad (5.4)$$

see [18] for more details.

Since (5.3) does not contain the other components of the spin, the operator  $\hat{s}_z$  commutes with the Hamiltonian  $\hat{H}$  and the  $z$ -component of the spin is conserved. Thus the operator  $\hat{s}_z$  can be replaced by its eigenvalue  $s_z = \sigma$  in the Hamiltonian (5.3), so we have

$$\hat{H} = \frac{1}{2m} \left( \hat{p}_x + \frac{eH}{c} y \right)^2 + \frac{1}{2m} \hat{p}_y^2 + \frac{1}{2m} \hat{p}_z^2 - \frac{\mu\sigma}{s} H - yF \quad (5.5)$$



with  $\sigma = -s, -s+1, \dots, s-1, s$ . Then the spin dependence of the wave function becomes insignificant and the wave function in the Schrödinger equation (5.2) can be taken as an ordinary coordinate function  $\Psi = \Psi(\mathbf{r}, t, \sigma)$ .

It should be noted that, the Hamiltonian (5.5) does not contain the coordinates  $x$  and  $z$  explicitly. Therefore the operators  $\hat{p}_x$  and  $\hat{p}_z$  also commute with the Hamiltonian and the  $x$  and  $z$  components of the linear momentum are conserved. The corresponding eigenvalues  $p_x$  and  $p_z$  take on all real values from  $-\infty$  to  $\infty$ . In this Letter we consider the case when the magnetic field  $H$  and the electric force  $F$  are arbitrary functions of time  $t$  only. Using the substitution

$$\Psi(\mathbf{r}, t) = e^{i(xp_x + zp_z - S_F(t))/\hbar} \psi(y, t) \quad (5.6)$$

with

$$\frac{dS_F}{dt} = \frac{p_z^2}{2m} - \frac{\mu\sigma}{s} H(t) + \frac{cp_x}{e} \frac{F(t)}{H(t)} \quad (5.7)$$

results in the one-dimensional Schrödinger equation of the harmonic oscillator driven by an external force in the  $y$ -direction

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial y^2} + \frac{m\omega_H^2}{2} (y - y_0)^2 \psi - F(t) (y - y_0) \psi \quad (5.8)$$

with time-dependent quantities

$$\omega_H(t) = \frac{|e| H(t)}{mc}, \quad y_0(t) = -\frac{cp_x}{eH(t)}, \quad a_H(t) = \sqrt{\frac{\hbar}{m\omega_H(t)}}. \quad (5.9)$$

When the magnetic field is a constant  $H' \equiv 0$ , the solution is well known (see [20], [21]). In the absence of external electric force  $F \equiv 0$ , there exist discrete energy values corresponding to motion in a plane perpendicular to the magnetic field, namely, the *Landau levels*. See [20], [21] and section 7 for more details.

In general, the following substitution

$$\psi(y, t) = \chi(\eta, t), \quad \eta = \frac{y - y_0(t)}{a_H(t)} \quad (5.10)$$

gives the time-dependent Schrödinger equation of the form (2.1) as follows,

$$i \frac{\partial \chi}{\partial t} = \frac{\omega_H(t)}{2} \left( -\frac{\partial^2}{\partial \eta^2} + \eta^2 \right) \chi - f(t) \eta \chi + i(g(t) + h(t) \eta) \frac{\partial \chi}{\partial \eta}, \quad (5.11)$$

where

$$f(t) = \frac{a_H(t) F(t)}{\hbar}, \quad g(t) = \frac{y'_0(t)}{a_H(t)}, \quad h(t) = \frac{a'_H(t)}{a_H(t)}, \quad (5.12)$$

which can be solved by the method discussed in the previous sections. In this case

$$\tau(t) = \frac{\omega'_H(t)}{\omega_H(t)} + 2 \frac{a'_H(t)}{a_H(t)} = (\ln(\omega_H a_H^2))' = 0, \quad \sigma(t) = \frac{1}{4} \omega_H^2(t)$$

and the corresponding characteristic equation (2.13) coincides with the equation of motion for the classical oscillator with a time-dependent frequency

$$\mu_H'' + \omega_H^2(t) \mu_H = 0, \quad \mu_H(0) = 0, \quad \mu_H'(0) = \omega_H(0). \quad (5.13)$$

The fundamental solution of (5.11) is given by

$$\chi = G_H(\eta, \eta', t) = \frac{1}{\sqrt{2\pi i \mu_H(t)}} e^{i(\alpha_H(t)\eta^2 + \beta_H(t)\eta\eta' + \gamma_H(t)(\eta')^2 + \delta_H(t)\eta + \varepsilon_H(t)\eta' + \kappa_H(t))}, \quad (5.14)$$

where

$$\alpha_H(t) = \frac{1}{2\omega_H(t)} \frac{d}{dt} (\ln \mu_H(t)) = \frac{m}{2\hbar} a_H^2(t) \frac{d}{dt} (\ln \mu_H(t)), \quad (5.15)$$

$$\beta_H(t) = -\frac{1}{\mu_H(t)} \frac{a_H(t)}{a_H(0)} = -\frac{1}{\mu_H(t)} \sqrt{\frac{\omega_H(0)}{\omega_H(t)}} = -\frac{1}{\mu_H(t)} \sqrt{\frac{H(0)}{H(t)}}, \quad (5.16)$$

$$\gamma_H(t) = \frac{\omega_H(0)}{2} \left( \frac{1}{\mu_H(t) \mu'_H(t)} - \int_0^t \left( \frac{\omega_H(\tau)}{\mu'_H(\tau)} \right)^2 d\tau \right), \quad (5.17)$$

$$\delta_H(t) = \frac{a_H(t)}{\hbar \mu_H(t)} \left( \delta_F^{(0)}(t) + p_x \delta_H^{(1)}(t) \right), \quad (5.18)$$

$$\delta_F^{(0)}(t) = \int_0^t \mu_H(\tau) F(\tau) d\tau, \quad (5.19)$$

$$\delta_H^{(1)}(t) = \frac{mc}{e} \int_0^t \frac{\mu'_H(\tau) H'(\tau)}{H^2(\tau)} d\tau \quad (5.20)$$

$$= \frac{e}{|e|} - \frac{mc\mu'_H(t)}{eH(t)} - \frac{e}{mc} \int_0^t \mu_H(\tau) H(\tau) d\tau,$$

$$\varepsilon_H(t) = \frac{1}{ma_H(0)} \left( \varepsilon_F^{(0)}(t) + p_x \varepsilon_H^{(1)}(t) \right), \quad (5.21)$$

$$\begin{aligned} \varepsilon_F^{(0)}(t) &= \int_0^t \left( 1 - \frac{\mu_H(\tau) \mu'_H(\tau)}{\mu_H(t) \mu'_H(t)} \right) \frac{F(\tau)}{\mu'_H(\tau)} d\tau \\ &\quad + \frac{\hbar e^2}{m^2 c^2} \int_0^t \left( \frac{H(\tau)}{\mu'_H(\tau)} \right)^2 \delta_F^{(0)}(t) d\tau, \end{aligned} \quad (5.22)$$

$$\varepsilon_H^{(1)}(t) = -\frac{\delta_H^{(1)}(t)}{\mu_H(t) \mu'_H(t)} + \frac{\hbar e^2}{m^2 c^2} \int_0^t \left( \frac{H(\tau)}{\mu'_H(\tau)} \right)^2 \delta_H^{(1)}(\tau) d\tau, \quad (5.23)$$

$$\kappa_H(t) = \frac{1}{2\hbar m} \left( \kappa_F^{(0)}(t) + p_x \kappa_F^{(1)}(t) + p_x^2 \kappa_H^{(2)}(t) \right), \quad (5.24)$$

$$\begin{aligned} \kappa_F^{(0)}(t) &= \frac{1}{\mu_H(t) \mu'_H(t)} \left( \delta_F^{(0)}(t) \right)^2 - \int_0^t \left( \frac{\omega_H(\tau) \delta_F^{(0)}(t)}{\mu'_H(\tau)} \right)^2 d\tau \\ &\quad - 2 \int_0^t \frac{F(\tau) \delta_H^{(1)}(\tau)}{\mu'_H(\tau)} d\tau, \end{aligned} \quad (5.25)$$

$$\begin{aligned} \frac{1}{2} \kappa_F^{(1)}(t) &= \frac{\delta_F^{(0)}(t) \delta_H^{(1)}(t)}{\mu_H(t) \mu'_H(t)} - \int_0^t \left( \frac{\omega_H(\tau)}{\mu'_H(\tau)} \right)^2 \delta_F^{(0)}(t) \delta_H^{(1)}(t) d\tau \\ &\quad - \int_0^t \frac{F(\tau) \delta_H^{(1)}(\tau)}{\mu'_H(\tau)} d\tau, \end{aligned} \quad (5.26)$$

$$\kappa_H^{(2)}(t) = \frac{\left(\delta_H^{(1)}(t)\right)^2}{\mu_H(t) \mu_H'(t)} - \int_0^t \left( \frac{\omega_H(\tau) \delta_H^{(1)}(\tau)}{\mu_H'(\tau)} \right)^2 d\tau \quad (5.27)$$

as a special case of (2.16)–(2.22).

## 6. THE PROPAGATOR IN THREE DIMENSIONS

By the superposition principle, the fundamental solution  $\Psi = G(\mathbf{r}, \mathbf{r}', t)$  of the time-dependent Schrödinger equation of a particle with a spin in a uniform magnetic field

$$\left( i\hbar \frac{\partial}{\partial t} - \hat{H} \right) G(\mathbf{r}, \mathbf{r}', t) = 0 \quad (6.1)$$

is given by the double Fourier integral of the form

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', t) &= \frac{1}{(2\pi\hbar)^2 a_H(0)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i((x-x')p_x + (z-z')p_z - S_F(t, p_x, p_z))/\hbar} \\ &\quad \times G_H\left(\frac{y-y_0(t)}{a_H(t)}, \frac{y'-y_0(0)}{a_H(0)}, t\right) dp_x dp_z. \end{aligned} \quad (6.2)$$

Here we obtain from (5.7) that

$$S_F(t, p_x, p_z) = \frac{p_z^2}{2m}t + \frac{cp_x}{e} \int_0^t \frac{F(\tau)}{H(\tau)} d\tau - \frac{\mu\sigma}{s} \int_0^t H(\tau) d\tau. \quad (6.3)$$

By (5.14) the corresponding Green function is represented as

$$G_H(\eta, \eta', t) = \frac{1}{\sqrt{2\pi i \mu_H(t)}} e^{iS_H(\eta, \eta', t)} \quad (6.4)$$

with

$$S_H(\eta, \eta', t) = \alpha_H(t) \eta^2 + \beta_H(t) \eta \eta' + \gamma_H(t) (\eta')^2 + \delta_H(t) \eta + \varepsilon_H(t) \eta' + \kappa_H(t) \quad (6.5)$$

and

$$\eta = \frac{y - y_0(t)}{a_H(t)}, \quad \eta' = \frac{y' - y_0(0)}{a_H(0)},$$

where  $y_0$  is a linear function of  $p_x$  [see (5.9) for a definition of functions  $y_0(t)$  and  $a_H(t)$ ]. Next, we are given

$$\lim_{t \rightarrow 0^+} G(\mathbf{r}, \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}') = \delta(x - x') \delta(y - y') \delta(z - z') \quad (6.6)$$

as our initial data by the asymptotic relation (2.24) and the integral representation

$$\delta(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha\xi} d\xi \quad (6.7)$$

as the Dirac delta function.

The quadratic in (6.5) is also a quadratic polynomial in  $p_x$  given as:

$$S_H(\eta, \eta', t) = S_H^{(2)}(y, y', t) + p_x S_H^{(1)}(y, y', t) + p_x^2 S_H^{(0)}(t) \quad (6.8)$$

with coefficients

$$S_H^{(0)}(t) = \frac{c^2 \alpha_H(t)}{e^2 a_H^2(t) H^2(t)} + \frac{c^2 \beta_H(t)}{e^2 a_H(t) a_H(0) H(t) H(0)} + \frac{c^2 \gamma_H(t)}{e^2 a_H^2(0) H^2(0)} \quad (6.9)$$

$$+ \frac{c \delta_H^{(1)}(t)}{\hbar e \mu_H(t) H(t)} + \frac{c \varepsilon_H^{(1)}(t)}{m e a_H^2(0) H(0)} + \frac{\kappa_H^{(2)}(t)}{2 \hbar m},$$

$$S_H^{(1)}(y, y', t) = \left( \frac{2c \alpha_H(t)}{e a_H^2(t) H(t)} + \frac{c \beta_H(t)}{e a_H(t) a_H(0) H(0)} + \frac{\delta_H^{(1)}(t)}{\hbar \mu_H(t)} \right) y \quad (6.10)$$

$$+ \left( \frac{c \beta_H(t)}{e a_H(t) a_H(0) H(t)} + \frac{2c \gamma_H(t)}{e a_H^2(0) H(0)} + \frac{\varepsilon_H^{(1)}(t)}{m a_H^2(0)} \right) y'$$

$$+ \frac{c \delta_F^{(0)}(t)}{\hbar e \mu_H(t) H(t)} + \frac{c \varepsilon_F^{(0)}(t)}{m e a_H^2(0) H(0)} + \frac{\kappa_F^{(1)}(t)}{2 \hbar m},$$

and

$$S_H^{(2)}(y, y', t) = \frac{\alpha_H(t)}{a_H^2(t)} y^2 + \frac{\beta_H(t)}{a_H(t) a_H(0)} y y' + \frac{\gamma_H(t)}{a_H^2(0)} (y')^2 \quad (6.11)$$

$$+ \frac{\delta_F^{(0)}(t)}{\hbar \mu_H(t)} y + \frac{\varepsilon_F^{(0)}(t)}{m a_H^2(0)} y' + \frac{\kappa_F^{(0)}(t)}{2 \hbar m}.$$

See equations (5.9) and (5.15)–(5.27) for notation explanation.

Hence, the double Fourier integral (6.2) can be evaluated in terms of elementary functions. So, integration over  $p_z$  gives the free particle propagator of motion in the direction of magnetic field in the following fashion,

$$G_0(z - z', t) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} \exp \left( \frac{i}{\hbar} \left( (z - z') p_z - \frac{p_z^2}{2m} t \right) \right) dp_z \quad (6.12)$$

$$= \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left( \frac{i m (z - z')^2}{2 \hbar t} \right)$$

using the familiar elementary integral

$$\int_{-\infty}^{\infty} e^{i(az^2 + 2bz)} dz = \sqrt{\frac{\pi i}{a}} e^{-ib^2/a}, \quad (6.13)$$

see Refs. [3] and [30].

Similarly, integration over  $p_x$  yields

$$G(\mathbf{r}, \mathbf{r}', t) = \frac{G_0(z - z', t)}{2\pi \hbar a_H(0) \sqrt{2\mu_H(t) S_H^{(0)}(t)}} \exp \left( \frac{i\mu\sigma}{\hbar s} \int_0^t H(\tau) d\tau \right) \quad (6.14)$$

$$\times \exp \left( \frac{1}{4i\hbar^2 S_H^{(0)}(t)} \left( x - x' - \frac{c}{e} \int_0^t \frac{F(\tau)}{H(\tau)} d\tau \right)^2 \right)$$

$$\begin{aligned} & \times \exp \left( \frac{\left( S_H^{(1)}(y, y', t) \right)^2 - 4S_H^{(0)}(t) S_H^{(2)}(y, y', t)}{4iS_H^{(0)}(t)} \right) \\ & \times \exp \left( \frac{S_H^{(1)}(y, y', t)}{2i\hbar S_H^{(0)}(t)} \left( x - x' - \frac{c}{e} \int_0^t \frac{F(\tau)}{H(\tau)} d\tau \right) \right) \end{aligned}$$

with

$$a_H(0) = \sqrt{\frac{\hbar}{m\omega_H(0)}} = \sqrt{\frac{\hbar c}{|e|H(0)}}.$$

Moreover, the quadratic form in this expression for the 3D-propagator

$$Q(y, y', t) = \left( S_H^{(1)}(y, y', t) \right)^2 - 4S_H^{(0)}(t) S_H^{(2)}(y, y', t) \quad (6.15)$$

(or the discriminant of (6.8)) can be rewritten as

$$Q(y, y', t) = A(t) y^2 + B(t) y y' + C(t) (y')^2 + D(t) y + E(t) y' + L(t), \quad (6.16)$$

where

$$\begin{aligned} A &= \frac{1}{2} \frac{\partial^2 Q}{\partial y^2}, & B &= \frac{\partial^2 Q}{\partial y \partial y'}, & C &= \frac{1}{2} \frac{\partial^2 Q}{\partial y'^2}, \\ D &= \left. \frac{\partial Q}{\partial y} \right|_{y=y'=0}, & E &= \left. \frac{\partial Q}{\partial y'} \right|_{y=y'=0}, & L &= Q|_{y=y'=0}. \end{aligned} \quad (6.17)$$

Our final result is

$$\begin{aligned} A(t) &= \frac{c^2 (\beta_H^2(t) - 4\alpha_H(t) \gamma_H(t))}{e^2 a_H^2(t) a_H^2(0) H^2(0)} + \frac{2c\beta_H(t) \delta_H^{(1)}(t)}{\hbar e \mu_H(t) a_H(t) a_H(0) H(0)} \\ &+ \frac{\left( \delta_H^{(1)}(t) \right)^2}{\hbar^2 \mu_H^2(t)} - \frac{4c\alpha_H(t) \varepsilon_H^{(1)}(t)}{m e a_H^2(t) a_H^2(0) H(0)} - \frac{2\alpha_H(t) \kappa_H^{(2)}(t)}{\hbar m a_H^2(t)}, \end{aligned} \quad (6.18)$$

$$\begin{aligned} B(t) &= -\frac{2c^2 (\beta_H^2(t) - 4\alpha_H(t) \gamma_H(t))}{e^2 a_H^2(t) a_H^2(0) H^2(0)} - \frac{2c\beta_H(t) \delta_H^{(1)}(t)}{\hbar e \mu_H(t) a_H(t) a_H(0) H(0)} \\ &+ \frac{4c\alpha_H(t) \varepsilon_H^{(1)}(t)}{m e a_H^2(t) a_H^2(0) H(t)} + \frac{4c\gamma_H(t) \delta_H^{(1)}(t)}{\hbar e \mu_H(t) a_H^2(0) H(0)} + \frac{2\delta_H^{(1)}(t) \varepsilon_H^{(1)}(t)}{\hbar m \mu_H(t) a_H^2(0)} \\ &- \frac{2c\beta_H(t) \varepsilon_H^{(1)}(t)}{m e a_H(t) a_H^3(0) H(0)} - \frac{2\beta_H(t) \kappa_H^{(2)}(t)}{\hbar m a_H(t) a_H(0)}, \end{aligned} \quad (6.19)$$

$$\begin{aligned} C(t) &= \frac{c^2 (\beta_H^2(t) - 4\alpha_H(t) \gamma_H(t))}{e^2 a_H^2(t) a_H^2(0) H^2(t)} + \frac{2c\beta_H(t) \varepsilon_H^{(1)}(t)}{m e a_H(t) a_H^3(0) H(t)} \\ &+ \frac{\left( \varepsilon_H^{(1)}(t) \right)^2}{m^2 a_H^4(0)} - \frac{4c\gamma_H(t) \delta_H^{(1)}(t)}{\hbar e \mu_H(t) a_H^2(0) H(t)} - \frac{4\gamma_H(t) \kappa_H^{(2)}(t)}{2\hbar m a_H^2(0)}, \end{aligned} \quad (6.20)$$

$$\begin{aligned}
D(t) = & \frac{4c^2\alpha_H(t)\varepsilon_F^{(0)}(t)}{me^2a_H^2(t)a_H^2(0)H(t)H(0)} + \frac{2c^2\beta_H(t)\varepsilon_F^{(0)}(t)}{me^2a_H(t)a_H^3(0)H^2(0)} \\
& + \frac{2c\left(\delta_H^{(1)}(t)\varepsilon_F^{(0)}(t) - 2\delta_F^{(0)}(t)\varepsilon_H^{(1)}(t)\right)}{\hbar me\mu_H(t)a_H^2(0)H(0)} + \frac{2c\alpha_H(t)\kappa_F^{(1)}(t)}{\hbar mea_H^2(t)H(t)} \\
& + \frac{c\beta_H(t)\kappa_F^{(1)}(t)}{\hbar mea_H(t)a_H(0)H(0)} + \frac{\delta_H^{(1)}(t)\kappa_F^{(1)}(t) - 2\delta_F^{(0)}(t)\kappa_H^{(2)}(t)}{\hbar^2 m\mu_H(t)} \\
& - \frac{2c^2\beta_H(t)\delta_F^{(0)}(t)}{\hbar e^2\mu_H(t)a_H(t)a_H(0)H(t)H(0)} - \frac{4c^2\gamma_H(t)\delta_F^{(0)}(t)}{\hbar e^2\mu_H(t)a_H^2(0)H^2(0)} \\
& - \frac{2c\delta_F^{(0)}(t)\delta_H^{(1)}(t)}{\hbar^2 e\mu_H^2(t)H(t)},
\end{aligned} \tag{6.21}$$

$$\begin{aligned}
E(t) = & \frac{2c^2\beta_H(t)\delta_F^{(0)}(t)}{\hbar e^2\mu_H(t)a_H(t)a_H(0)H^2(t)} + \frac{4c^2\gamma_H(t)\delta_F^{(0)}(t)}{\hbar e^2\mu_H(t)a_H^2(0)H(t)H(0)} \\
& + \frac{2c\left(\delta_F^{(0)}(t)\varepsilon_H^{(1)}(t) - 2\delta_H^{(1)}(t)\varepsilon_F^{(0)}(t)\right)}{\hbar me\mu_H(t)a_H^2(0)H(t)} + \frac{c\beta_H(t)\kappa_F^{(1)}(t)}{\hbar mea_H(t)a_H(0)H(t)} \\
& + \frac{2c\gamma_H(t)\kappa_F^{(1)}(t)}{\hbar mea_H^2(0)H(0)} + \frac{\varepsilon_H^{(1)}(t)\kappa_F^{(1)}(t) - 2\varepsilon_F^{(0)}(t)\kappa_H^{(2)}(t)}{\hbar m^2a_H^2(0)} \\
& - \frac{4c^2\alpha_H(t)\varepsilon_F^{(0)}(t)}{me^2a_H^2(t)a_H^2(0)H^2(t)} - \frac{2c^2\beta_H(t)\varepsilon_F^{(0)}(t)}{me^2a_H(t)a_H^3(0)H(t)H(0)} \\
& - \frac{2c\delta_F^{(0)}(t)\delta_H^{(1)}(t)}{m^2ea_H^4(0)H(0)},
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
L(t) = & \frac{c^2\left(\delta_F^{(0)}(t)\right)^2}{\hbar^2 e^2\mu_H^2(t)H^2(t)} + \frac{c^2\left(\varepsilon_F^{(0)}(t)\right)^2}{m^2 e^2a_H^4(0)H^2(0)} + \frac{\left(\kappa_F^{(1)}(t)\right)^2}{4\hbar^2 m^2} \\
& + \frac{2c^2\delta_F^{(0)}(t)\varepsilon_F^{(0)}(t)}{\hbar me^2\mu_H(t)a_H^2(0)H(t)H(0)} + \frac{c\delta_F^{(0)}(t)\kappa_F^{(1)}(t)}{\hbar^2 me\mu_H(t)H(t)} + \frac{c\varepsilon_F^{(0)}(t)\kappa_F^{(1)}(t)}{\hbar m^2ea_H^2(0)H(0)} \\
& - \frac{2c^2\alpha_H(t)\kappa_F^{(0)}(t)}{\hbar me^2a_H^2(t)H^2(t)} - \frac{2c^2\beta_H(t)\kappa_F^{(0)}(t)}{\hbar me^2a_H(t)a_H(0)H(t)H(0)} - \frac{2c^2\gamma_H(t)\kappa_F^{(0)}(t)}{\hbar me^2a_H^2(0)H^2(0)} \\
& - \frac{2c\delta_H^{(1)}(t)\kappa_F^{(0)}(t)}{\hbar^2 me\mu_H(t)H(t)} - \frac{2c\varepsilon_H^{(1)}(t)\kappa_F^{(0)}(t)}{\hbar m^2ea_H^2(0)H(0)} - \frac{\kappa_F^{(0)}(t)\kappa_H^{(2)}(t)}{\hbar^2 m^2}.
\end{aligned} \tag{6.23}$$

The details are left to the reader.

Finally, by the superposition principle, a general solution of the Cauchy initial value problem in  $\mathbf{R}^3$  subject to the initial data

$$\Psi(\mathbf{r}, t)|_{t=0} = \Psi(\mathbf{r}, 0) = \phi(x, y, z) \tag{6.24}$$

has the form

$$\Psi(\mathbf{r}, t) = \int_{\mathbf{R}^3} G(\mathbf{r}, \mathbf{r}', t) \Psi(\mathbf{r}', 0) dx' dy' dz'. \quad (6.25)$$

This gives us the time evolution operator (1.2) explicitly for a motion of a charged particle in a uniform magnetic field and also a perpendicular electric field with a given projection of the spin  $s_z = \sigma$  in the direction of magnetic field.

## 7. TWO EXAMPLES

**7.1. Motion in a Constant Magnetic Field.** The simplest case occurs as  $H' = F = 0$ , which implies  $\mu_H'' + \omega_H^2 \mu_H = 0$  with  $\mu_H = \sin(\omega_H t)$ . Thus,

$$\begin{aligned} \alpha_H(t) &= \gamma_H(t) = \frac{1}{2} \cot(\omega_H t), \quad \beta_H(t) = -\frac{1}{\sin(\omega_H t)}, \\ \delta_F^{(0)} &= \delta_H^{(1)} = \varepsilon_F^{(0)} = \varepsilon_H^{(1)} = \kappa_F^{(0)} = \kappa_F^{(1)} = \kappa_H^{(2)} = 0, \end{aligned} \quad (7.1)$$

and

$$\begin{aligned} S_H^{(0)}(t) &= -\frac{1}{m\hbar\omega} \tan\left(\frac{\omega_H t}{2}\right), \quad S_H^{(1)}(y, y', t) = -\frac{1}{\hbar|e|} \tan\left(\frac{\omega_H t}{2}\right) (y + y'), \\ S_H^{(2)}(y, y', t) &= \frac{m\omega}{2\hbar} \left( \cot(\omega_H t) y^2 - \frac{2yy'}{\sin(\omega_H t)} + \cot(\omega_H t) (y')^2 \right), \end{aligned} \quad (7.2)$$

where the discriminant is

$$Q(y, y') = \left( S_H^{(1)}(y, y', t) \right)^2 - 4S_H^{(0)}(t) S_H^{(2)}(y, y', t) = \frac{1}{\hbar^2} (y - y')^2. \quad (7.3)$$

Hence, the Green function is given by

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', t) &= G_0(z - z', t) \exp\left(\frac{i\mu\sigma H t}{\hbar s}\right) \frac{m\omega_H}{4\pi i \hbar \sin(\omega_H t/2)} \\ &\times \exp\left(\frac{im\omega_H}{4\hbar} \left( ((x - x')^2 + (y - y')^2) \cot(\omega_H t/2) - 2\frac{e}{|e|} (x - x')(y + y') \right)\right). \end{aligned} \quad (7.4)$$

See [21] for more details.

**7.2. A Linear Magnetic Field.** Now consider the case  $H(t) = H_0 + tH_1$ , where  $H_0$  and  $H_1$  are constants. The characteristic equation (5.13) becomes a special case of the Lommel equation [1], [28], [38], which can be solved in terms of Bessel functions [40] of orders  $\nu = \pm 1/4$ . It then follows that

$$\begin{aligned} \mu_H(t) &= \frac{\pi|e|H_0^{3/2}}{2^{3/2}mcH_1} \sqrt{H(t)} \\ &\times \left( J_{-1/4}\left(\frac{|e|H_0^2}{2mcH_1}\right) J_{1/4}\left(\frac{|e|H^2(t)}{2mcH_1}\right) - J_{1/4}\left(\frac{|e|H_0^2}{2mcH_1}\right) J_{-1/4}\left(\frac{|e|H^2(t)}{2mcH_1}\right) \right) \end{aligned} \quad (7.5)$$

with

$$\frac{d\mu_H(t)}{dt} = \frac{\pi e^2}{m^2 c^2 H_1} \left( \frac{H_0 H(t)}{2} \right)^{3/2} \quad (7.6)$$

$$\times \left( J_{1/4} \left( \frac{|e| H_0^2}{2mcH_1} \right) J_{3/4} \left( \frac{|e| H^2(t)}{2mcH_1} \right) + J_{-1/4} \left( \frac{|e| H_0^2}{2mcH_1} \right) J_{-3/4} \left( \frac{|e| H^2(t)}{2mcH_1} \right) \right).$$

A general expression of the propagator is given above by (6.14). In the case when  $F \equiv 0$ , one can simplify  $\delta_F^{(0)} = \varepsilon_F^{(0)} = \kappa_F^{(0)} = \kappa_F^{(1)} \equiv 0$ . We shall elaborate on these and some other interesting special cases elsewhere.

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